Backreaction for Einstein-Rosen waves coupled to a massless scalar field

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Introduction

- SJS, M. Wyrębowski, PRD 94, 024059 (2016)
- Exact solutions as a toy-models for studies of backreaction
- Standard averaging procedure (averaging of the metric)

$$A(g_U) = g^{(0)}$$

A — averaging operator
g_U — the metric describing spacetime with small scale inhomogeneities
$g^{(0)}$ — the effective metric

Natural assumptions on A:

1. A is covariant
2. A is unique
3. $A^2 = A$
4. ...

an example: Charach, Malin PRD 19:1058 (1979)

- Gowdy T^3 cosmology $\longleftrightarrow$ Einstein-Rosen waves (almost 1 : 1)
- A trick to interpret new solutions

$$g_U = g^{(0)} + h$$

this split is coordinate dependent!
g_U — the metric describing spacetime with small scale inhomogeneities
$g^{(0)}$ — the effective metric
h — fast oscillating component of the metric g_U

Calculate $\hat{T} = \frac{1}{8\pi} G(g^{(0)})$

- the Charach, Malin miracle: $\hat{T}$ has a nice interpretation (double null dust)

$$G(g_U) = 8\pi\alpha T$$

T — massless scalar field

$$\hat{T} = \alpha T^{(0)} + t^{(0)}$$

$T^{(0)}$, $t^{(0)}$ — null fluids

an example: Charach, Malin 1979

- Charach, Malin: an another way to calculate $T^{(0)}$: $T^{(0)} =< T >$

Alternative procedure to study backreaction

- guess a simplified metric
- study how it violates Einstein equations assuming the knowledge of the original energy content

an example: Charach, Malin 1979

- Charach, Malin: an another way to calculate $T^{(0)}$: $T^{(0)} =< T >$

Alternative procedure to study backreaction

- guess a simplified metric
- study how it violates Einstein equations assuming the knowledge of the original energy content

- Three frameworks to calculate $t^{(0)}$ (backreaction)
  1. Charach, Malin paper
  2. generalized Isaacson procedure
  3. Green and Wald framework
- Three frameworks (they differ at the technical level) give the same backreaction term

The G-W assumptions

(i)

There exists a one-parameter family $g_{ab}(\lambda)$, $\lambda > 0$, satisfying

$$G_{ab}(g(\lambda)) = 8\pi \, T_{ab}(\lambda) \, , \quad \lambda > 0 \, ,$$

where $T_{ab}(\lambda) t^a(\lambda) t^b(\lambda) \geq 0$ for all timelike $t^a(\lambda)$.

(ii)

There exists a smooth background metric

$$g^{(0)}_{ab} := \lim_{\lambda \to 0} g_{ab}(\lambda)$$

$$h_{ab}(\lambda) := g_{ab}(\lambda) - g^{(0)}_{ab} \to 0 \quad \text{for} \quad \lambda \to 0 \, .$$

The G-W assumptions

(iii)

Derivatives $\nabla_a h_{bc}(\lambda)$ are *bounded* in the limit $\lambda \to 0$ (do not necessarily go to zero).
No assumptions about second derivatives – can be unbounded.

(iv)

There exists a smooth tensor field

$$\mu_{abcdef} := \underset{\lambda \to 0}{\text{w-lim}} \left[\nabla_a h_{cd}(\lambda) \nabla_b h_{ef}(\lambda) \right] .$$

The weak limit

We say that a one-parameter family of smooth tensor fields $A_{a_1 \ldots a_n}(\lambda)$, $\lambda > 0$ converges *weakly* to $B_{a_1 \ldots a_n}$ as $\lambda \to 0$,

$$B_{a_1 \ldots a_n} = \underset{\lambda \to 0}{\text{w-lim}}\, A_{a_1 \ldots a_n}(\lambda)\,,$$

if for all smooth $f^{a_1 \cdots a_n}$ of compact support, we have

$$\lim_{\lambda \to 0} \int f^{a_1 \ldots a_n} A_{a_1 \ldots a_n}(\lambda) = \int f^{a_1 \ldots a_n} B_{a_1 \ldots a_n}\,.$$

Equation satisfied by the background metric

$$G_{ab}(g^{(0)}) = 8\pi\, T^{(0)}_{ab} + 8\pi\, t^{(0)}_{ab}\,.$$

where

- $T^{(0)}_{ab} := \text{w-lim}_{\lambda\to 0}\,(T_{ab}(\lambda))$ – the 'averaged matter energy-momentum tensor'
- $t^{(0)}_{ab}$ – the 'effective gravitational energy-momentum tensor', contribution from nonlinear (second order in $h(\lambda)$) terms in the Einstein tensor:

The effective energy-momentum tensor

$$8\pi\, t^{(0)}_{ab} = \frac{1}{8}\left(-\mu^{c}{}_{c}{}^{de}{}_{de} - \mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + 2\,\mu^{cd}{}_{c}{}^{e}{}_{de}\right) g^{(0)}_{ab} + \frac{1}{2}\,\mu^{cd}{}_{acbd} - \frac{1}{2}\,\mu^{c}{}_{ca}{}^{d}{}_{bd} + \frac{1}{4}\,\mu_{ab}{}^{cd}{}_{cd} - \frac{1}{2}\,\mu^{c}{}_{(ab)c}{}^{d}{}_{d} + \frac{3}{4}\,\mu^{c}{}_{cab}{}^{d}{}_{d} - \frac{1}{2}\,\mu^{cd}{}_{abcd}$$

Theorem (Green and Wald, 2011)

$t^{(0)}_{ab}$ is traceless and satisfies the weak energy condition.

The Einstein-Rosen metric

The line element has the form (Einstein, Rosen 1937; Rosen 1954)

$$g = \mathsf{e}^{2(\gamma-\psi)}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right) + \rho^2\,\mathsf{e}^{-2\psi}\,\mathrm{d}\varphi^2 + \mathsf{e}^{2\psi}\,\mathrm{d}z^2\,,$$

where (cylindrical symmetry)

$$\begin{aligned} &\rho > 0, \quad -\infty < t, z < \infty, \quad 0 \leq \varphi < 2\pi, \\ &\psi = \psi(t,\rho), \qquad \gamma = \gamma(t,\rho)\,. \end{aligned}$$

Include massless minimally coupled scalar field ϕ to obtain nonvacuum solutions.

The Einstein-Rosen-scalar field solution

Field equations reduce to

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0,$$
$$\gamma' = \rho\left(\dot{\phi}^2 + \phi'^2 + \dot{\psi}^2 + \psi'^2\right),$$
$$\dot{\gamma} = 2\rho\left(\dot{\phi}\phi' + \dot{\psi}\psi'\right),$$
$$\phi'' + \frac{1}{\rho}\phi' - \ddot{\phi} = 0.$$

The energy density of the scalar field as measured by observers comoving with the coordinate system (with four-velocity $u = e^{\psi-\gamma}\partial_t$)

$$\epsilon = T_{ab}u^a u^b = \frac{1}{8\pi}\, e^{2(\psi-\gamma)}\left(\dot{\phi}^2 + \phi'^2\right).$$

One-parameter family of solutions

We choose the following particular solutions of the field equations:

$$\phi_\lambda(t,\rho) = \alpha\sqrt{\lambda}\, F_\lambda(t,\rho)\,, \quad \psi_\lambda(t,\rho) = \beta\sqrt{\lambda}\, F_\lambda(t,\rho)\,, \quad \lambda > 0\,,$$

where: $F_\lambda(t,\rho) = J_0\left(\frac{\rho}{\lambda}\right)\sin\left(\frac{t}{\lambda}\right)$; λ – parameter; J_0 – Bessel function of the first kind and zero order; constants α, β – real and independent of λ.
Integrating the remaining field equations we get

$$\gamma_\lambda(t,\rho) = \frac{(\alpha^2+\beta^2)}{2\lambda}\rho^2\left[J_0^2(\frac{\rho}{\lambda}) + J_1^2(\frac{\rho}{\lambda}) - 2\frac{\lambda}{\rho}J_0(\frac{\rho}{\lambda})J_1(\frac{\rho}{\lambda})\sin^2(\frac{t}{\lambda})\right].$$

This gives $g(\lambda)$.

One-parameter family of solutions

Let $A_\lambda(t,\rho) = J_0\left(\frac{\rho}{\lambda}\right)\cos\left(\frac{t}{\lambda}\right)$ and $B_\lambda(t,\rho) = J_1\left(\frac{\rho}{\lambda}\right)\sin\left(\frac{t}{\lambda}\right)$. Then for $\lambda > 0$ the nonzero components of $T_{ab}(\lambda)$ are

$$
\begin{aligned}
T_{tt}(\lambda) &= T_{\rho\rho}(\lambda) = \frac{\alpha^2}{8\pi\lambda}(A_\lambda^2 + B_\lambda^2)\,, \\
T_{t\rho}(\lambda) &= T_{\rho t}(\lambda) = -\frac{\alpha^2}{4\pi\lambda}A_\lambda B_\lambda\,, \\
T_{\varphi\varphi}(\lambda) &= \frac{\alpha^2}{8\pi\lambda}\,\mathrm{e}^{-2\gamma_\lambda}\,\rho^2(A_\lambda^2 - B_\lambda^2)\,, \\
T_{zz}(\lambda) &= T_{\varphi\varphi}(\lambda)\,\rho^{-2}\,\mathrm{e}^{4\beta\sqrt{\lambda}F_\lambda}\,.
\end{aligned}
$$

Then

$$
\epsilon(\lambda) = \frac{1}{8\pi}\frac{\alpha^2}{\lambda}\,\mathrm{e}^{2(\beta\sqrt{\lambda}F_\lambda - \gamma_\lambda)}(A_\lambda^2 + B_\lambda^2)\,.
$$

Note the nontrivial behavior in the limit $\lambda \to 0$.

The background metric

For $\rho/\lambda \gg 1$:

$$J_n(\frac{\rho}{\lambda}) = \sqrt{\frac{2}{\pi}\frac{\lambda}{\rho}}\left[\cos(\rho/\lambda - \frac{\pi}{2}n - \frac{\pi}{4}) + O(\frac{\lambda}{\rho})\right] .$$

Using this we find the limit as $\lambda \to 0$:

$$\psi_\lambda \to 0 \, , \quad \gamma_\lambda \to (\alpha^2 + \beta^2)\rho/\pi \, , \quad \phi_\lambda \to 0 \, .$$

Thus

$$g^{(0)} = e^{2(\alpha^2+\beta^2)\rho/\pi}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right) + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 \, .$$

The limiting functions do not satisfy one of the field equations, hence $g^{(0)}$ does not belong to the described class of solutions.

The background metric

The nonzero components of $G_{ab}(g^{(0)})$ are

$$G_{tt}(g^{(0)}) = G_{\rho\rho}(g^{(0)}) = \frac{\alpha^2 + \beta^2}{\pi\rho}\,.$$

In the weak limit the nonzero components of the scalar field energy-momentum tensor are

$$T_{tt}^{(0)} = T_{\rho\rho}^{(0)} = \frac{\alpha^2}{8\pi^2\rho}\,.$$

So, the effective energy-momentum tensor can be readily calculated:
$t_{ab}^{(0)} = \frac{1}{8\pi} G_{ab}(g^{(0)}) - T_{ab}^{(0)}$.

The μ tensor

$$
\begin{aligned}
&\mu_{tttttt} = \mu_{tt\rho\rho\rho\rho} = -\mu_{tttt\rho\rho} \\
&= \mu_{\rho\rho tttt} = \mu_{\rho\rho\rho\rho\rho\rho} = -\mu_{\rho\rho tt\rho\rho} = \left[\frac{2}{\pi}\beta^2\rho^{-1} + \frac{1}{\pi^2}\left(\alpha^2+\beta^2\right)^2\right] e^{4(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\varphi\varphi\varphi\varphi} = \mu_{\rho\rho\varphi\varphi\varphi\varphi} = \frac{2}{\pi}\beta^2\rho^3\,, \\
&\mu_{ttzzzz} = \mu_{\rho\rho zzzz} = \frac{2}{\pi}\beta^2\rho^{-1}\,, \\
&\mu_{tt\rho\rho\varphi\varphi} = -\mu_{tttt\varphi\varphi} = \mu_{\rho\rho\rho\rho\varphi\varphi} = -\mu_{\rho\rho tt\varphi\varphi} = \frac{2}{\pi}\beta^2\rho\, e^{2(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\rho\rho zz} = -\mu_{ttttzz} = \mu_{\rho\rho\rho\rho zz} = -\mu_{\rho\rho ttzz} = -\frac{2}{\pi}\beta^2\rho^{-1} e^{2(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\varphi\varphi zz} = \mu_{\rho\rho\varphi\varphi zz} = -\frac{2}{\pi}\beta^2\rho\,.
\end{aligned}
$$

All other components follow from $\mu_{abcdef} = \mu_{(ab)(cd)(ef)} = \mu_{abefcd}$ or vanish.

The effective energy-momentum tensor

Nonzero components:

$$t_{tt}^{(0)} = t_{\rho\rho}^{(0)} = \frac{\beta^2}{8\pi^2 \rho}\,.$$

Properties:

- $g(\lambda)$ satisfies all GW assumptions
- $t^{(0)}$ satisfies all GW theorems (traceless, WEC)
- Inhomogeneities of the scalar field do not contribute in the leading order to the backreaction effect (no dependence on α).
- For the chosen subclass of solutions $t^{(0)}$ is unique (path independent, as in all remaining known examples)